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Sequences with a Unique Realization by Simple Graphs*

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Sequences with unique realizations (up to isomorphism) by simple graphs are characterized, partly by reduction to the case of pairs of sequences, which have a unique realization by bipartite graphs (see Theorem 6.1).

1. INTRODUCTION

All graphs in this paper are finite, undirected, and without loops or multiple edges.

$G(p_1, ..., p_n)$ denotes a graph with n vertices $p_1, ..., p_n$. If two vertices p, q are adjacent in G, we write $(p, q) \in G$.

$d(p_i)$ denotes the degree of p_i in G.

A sequence $\psi = (b_1, ..., b_n)$ is *graphic* if there exists a graph $G = G(p_1, ..., p_n)$ such that $d(p_i) = a_i$ for $i = 1, ..., n$. G is called a *realization* of ψ. We say also that G *realizes* ψ. If every two realizations of ψ are isomorphic we call ψ *unigraphic*.

Similarly, a pair of sequences $[(b_1, ..., b_n), (c_1, ..., c_m)]$ is graphic if $d(p_i) = b_i$ $(i = 1, ..., n)$ and $d(q_i) = c_i$ $(i = 1, ..., m)$ for some bipartite graph $G(p_1, ..., p_n, q_1, ..., q_m)$, all its edges are of the form (p_{i_1}, q_{i_2}).

A graphic pair is *unigraphic* if any two realizations of it are isomorphic.

In this paper we characterize unigraphic sequences. One of the criteria for unigraphicness of a sequence is the unigraphicness of a pair of sequences. A characterization of unigraphic pairs was given in [5].

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In [2], Erdös and Gallai show that a nonincreasing sequence $(b_1, ..., b_n)$ of natural numbers is graphic iff:

$$\sum_{i=1}^{n} b_i \text{ is even} \tag{1.1}$$

$$\sum_{i=1}^{j} b_i - j(j-1) \leqslant \sum_{i=j+1}^{n} \min(j, b_i), \qquad (j = 1, ..., n). \tag{1.2}$$

Notation. Throughout this paper ϕ will denote the graphic nonincreasing sequence $(a_1, ..., a_n)$ of positive integers, and a, with some lower index, will denote a term of ϕ.

DEFINITION 1.1. *For a nonincreasing sequence $\psi = (b_1, ..., b_m)$ we define*

$$\delta(j, \psi) = \sum_{i=j+1}^{m} \min(j, b_i) - \sum_{i=1}^{j} b_i + j(j-1) \qquad (\text{for } j = 0, 1, ..., m).$$

Remark. Of course, $\delta(0, \psi) = 0$ for any ψ. $\delta(j, \phi) \geqslant 0$ for $j = 1, ..., n$; $\delta(n, \phi) = 0$ iff $a_1 = a_n = n - 1$, otherwise $\delta(n, \phi) > 1$.

As always, $K = K_n$ denotes the complete graph on n vertices. $K^c = K_n{}^c$ denotes the empty graph on n vertices; i.e., the graph with no edges.

Similarly, $K = K_{n,m}$ denotes the complete bipartite graph with two independent sets having n and m vertices, respectively. $K^c = K_{n,m}^c$ denotes the empty graph on $n + m$ vertices.

If X and Y are sets of vertices of G, $G[X, Y]$ denotes the subgraph of G, generated by X, Y, i.e., the graph with $X \cup Y$ as its set of vertices, which includes exactly those edges of G having one endpoint in X and one endpoint in Y. We write $G[X]$ for $G[X, X]$.

2. THE STRUCTURE OF SOME GRAPHS

LEMMA 2.1. *Suppose $\delta(j, \phi) = 0$ for some j, $1 \leqslant j < n$. If $a_{j+1} > j$, let k be an index such that $a_k \geqslant j \geqslant a_{k+1}$. If $a_{j+1} \leqslant j$, let $k = j$. For any realization $G = G\{p_1, ..., p_n\}$ of ϕ, define $S = \{p_1, ..., p_j\}$, $T = \{p_{k+1}, ..., p_n\}$, $U = \{p_{j+1}, ..., p_k\}$. Then:*

$$G[S] = K \tag{2.1}$$

$$G[T] = K^c \tag{2.2}$$

and if $U \neq \varnothing$

$$G[S, U] = K \tag{2.3}$$

$$G[T, U] = K^c. \tag{2.4}$$

Proof. $\sum_{i=1}^{j} a_i - j(j-1)$ and $\sum_{i=j+1}^{n} \min(j, a_i)$ are lower and upper bounds, respectively, on the number of edges of $G[S, T \cup U]$; since they are equal there must be $\frac{1}{2}j(j-1)$ edges in $G[S]$, and $\min(j, a_i)$ edges in $G[S, \{p_i\}]$ for $i > j$; and the results follow (see Fig. 1).

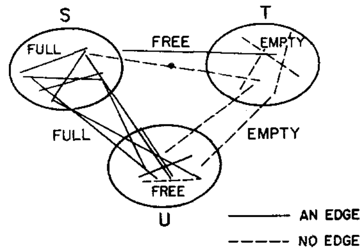

FIG. 1. The structure of G, when $\delta(j, \phi) = 0$.

The converse Lemma is also true:

LEMMA 2.2. *Suppose $G(p_1, \ldots, p_n)$ realizes ϕ, $S = \{p_1, \ldots, p_j\}$, $n > j \geqslant 1$, $T = \{p_{k+1}, \ldots, p_n\}$, $k \geqslant j$, $U = \{p_{j+1}, \ldots, p_k\}$ and conditions* (2.1), (2.2) *hold for S and T and, if $U \neq \varnothing$, conditions* (2.3), (2.4) *hold as well. Then* $\delta(j, \phi) = 0$.

Proof. The proof is straightforward and is omitted (see Fig. 1).

DEFINITION 2.1. $h = h(j) = \max\{j, \max\{i\colon a_i > j\}\}$; i.e., $h = j$ if $a_{j+1} \leqslant j$, $h = n$ if $a_n > j$, and if $a_{j+1} > j \geqslant a_n$, then h is defined by $a_h > j \geqslant a_{h+1}$.

LEMMA 2.3. *Suppose $a_1 = a_2 = \cdots = a_l > a_{l+1} \geqslant \cdots \geqslant a_n$ $(1 \leqslant l \leqslant n)$, $\delta(j, \phi) = 1$ for some j, $1 \leqslant j < l$ and $j \geqslant a_n$. Then $\delta(j+1, \phi) \leqslant 1$.*

Proof. As $1 + \sum_{i=1}^{j} a_i - j(j-1) = \sum_{i=j+1}^{n} \min(j, a_i)$, we must have $a_{j+1} > j$, since otherwise

$$1 + \sum_{i=1}^{j} a_i - j(j-1) = \sum_{i=j+1}^{n} a_i,$$

a contradiction of condition (1.1). Hence $n > h > j$. Thus

$$1 + \sum_{i=1}^{j} a_i = j(j-1) + j(h-j) + \sum_{i=h+1}^{n} a_i. \tag{2.5}$$

Hence

$$ja_1 \geqslant j(h-1), \qquad \text{or} \qquad a_1 \geqslant h-1.$$

Suppose

$$1 + \sum_{i=1}^{j+1} a_i < (j+1)j + (j+1)(h-j-1) + \sum_{i=h+1}^{n} a_i . \quad (2.6)$$

Subtracting (2.5) from (2.6) we obtain $a_1 = a_{j+1} < h - 1$, a contradiction.

LEMMA 2.4. *Suppose* $\delta(j, \phi) = 0$ *for some* j, $1 \leqslant j < n$, *then* ϕ *is unigraphic iff the pair* $[(a_1 - (h-1),..., a_j - (h-1)), (a_{h+1},..., a_n)]$ *is unigraphic and so is the sequence* $(a_{j+1} - j,..., a_h - j)$, (*if* $h > j$).

Proof. Lemma 2.4 follows easily from Lemma 2.1.

3. PROPERTIES OF UNIGRAPHIC SEQUENCES

We will need the following known Theorem:

THEOREM 3.1. ([2] *and compare* [3].) *If* $\psi = (b_1,..., b_n)$ *is graphic, then for each* j, $j = 1,..., n$, *there exists a realization* $G_j(p_1,..., p_n)$ *of* ψ *in which* p_j *is adjacent to the first* (*other*) b_j *vertices with the highest degrees.*

COROLLARY 3.1. *If* $a_1 > a_2 > a_{n-1} > a_n$, *then* ϕ *has a realization* $G = G(p_1,..., p_n)$, *such that* $(p_{n-1}, p_1), (p_2, p_1) \in G$ *and* (p_{n-1}, p_n) $(p_2, p_n) \notin G$.

Proof. By Theorem 3.1, $\psi^* = (a_1 - 1, a_3 - 1,..., a_{a_2+1} - 1, a_{a_2+2},..., a_n)$ is graphic. By the same theorem ψ^* has a realization $G_{n-1}(p_1, p_3,..., p_n)$, in which $(p_{n-1}, p_1) \in G_{n-1}$ and $(p_{n-1}, p_n) \notin G_{n-1}$. Adding the vertex p_2 and connecting it to $p_1, p_3, p_4,..., p_{a_2+1}$ we obtain the desired realization G.

DEFINITION 3.1. *If* $(p_{i_1}, p_{i_2}) \in G$ $((p_{i_1}, p_{i_2}) \notin G)$ *then* $G - (p_{i_1}, p_{i_2})$ $(G + (p_{i_1}, p_{i_2}))$ *denotes the graph which is obtained from* G *by removing* (*adding*) *the edge* (p_{i_1}, p_{i_2}).

The next Theorem is one of our main results.

THEOREM 3.2. *If* $a_2 > a_{n-1}$ $(n \geqslant 4)$ *and* ϕ *is unigraphic, then* $\delta(j, \phi) = 0$ *for some* j, $1 \leqslant j < n$.

Proof. Assume $\delta(j, \phi) > 0$ for $j = 1,..., n-1$. We will consider two cases.

Case a. $\delta(j, \phi) > 1$ for $j = 1,..., n-1$.

In this case $\phi^* = (a_1 + 1, a_2, \ldots, a_{n-1}, a_n - 1)$ is graphic and, by Corollary 3.1, ϕ^* has a realization $G^* = G^*(p_1, \ldots, p_n)$ such that $(p_2, p_1), (p_{n-1}, p_1) \in G^*$ and $(p_2, p_n), (p_{n-1}, p_n) \notin G^*$. Thus $G_1 = G^* - (p_2, p_1) + (p_1, p_n)$ and $G_2 = G^* - (p_{n-1}, p_1) + (p_{n-1}, p_n)$ realize ϕ. Furthermore, G_1 and G_2 are not isomorphic since the number of edges between a vertex of degree a_1 and a vertex of degree a_2 is smaller in G_1 than in G_2.

Case b. $\delta(j, \phi) = 1$ for some values of j.

Let j_0 be the largest such j (i.e., $\delta(j_0, \phi) = 1$ and $\delta(j, \phi) > 1$ for $j > j_0$).

Since $\delta(j_0, \phi) = 1$ we have, by condition (1.1), $a_{j_0+1} > j_0$. If $h = h(j_0) = n$ (see Definition 2.1), then we proceed as in case a. Assume, therefore, that $n > h > j_0$. Then the sequence $\psi = (a_1 + 1, a_2, \ldots, a_{h-1}, a_h - 1, a_{h+1}, \ldots, a_n)$ is graphic and $\delta(j_0, \psi) = 0$. Since $a_n \leqslant j_0 < h$, ψ has, by Theorem 3.1, a realization $G' = G'(p_1, \ldots, p_n)$ such that $(p_n, p_1) \in G'$ and $(p_n, p_h) \notin G'$. Therefore $G = G' - (p_1, p_n) + (p_h, p_n)$ realizes ϕ and for $S = \{p_1, \ldots, p_{j_0}\}$, $T = \{p_{h+1}, \ldots, p_n\}$, $U = \{p_{j_0+1}, \ldots, p_h\}$; conditions (2.1) (2.2) and (2.3) hold, and $G[T, U] = K^c + (p_h, p_n)$.

If $h = j_0 + 1$, then, by Lemma 2.2, $\delta(j_0 + 1, \phi) = 0$, a contradiction. Hence $h > j_0 + 1$.

Since $G[T, U - \{p_h\}] = K^c$, we have

$$h - 1 \geqslant d(p_{j_0+1}) = a_{j_0+1} \geqslant a_h = d(p_h).$$

But $G[S, U] = K$, hence there exists an index r, $h > r > j_0$, such that $(p_r, p_h) \notin G$. Of course, $(p_r, p_1) \in G$, and thus another realization of ϕ is $G_1 = G' - (p_1, p_r) - (p_h, p_n) + (p_1, p_n) + (p_r, p_h)$. But $d(p_r) > d(p_n)$ and hence by the unigraphicness of ϕ we have $d(p_1) = d(p_h)$. Thus, by Lemma 2.3, we arrived at a contradiction to the maximality of j_0.

The next theorem sharpens the results of Theorem 3.2:

THEOREM 3.3. *Suppose ϕ is unigraphic, j_0 is the largest index j for which $\delta(j, \phi) = 0$ and $h' = h(j_0) > j_0$. Then $h' \geqslant j_0 + 3$, $a_{j_0+1} < h' - 1$ and $a_{j_0+2} = a_{h'-1}$.*

Proof. By Lemma 2.4, the sequence $\psi = (a_{j_0+1} - j_0, \ldots, a_{h'} - j_0) = (b_1, \ldots, b_m)$ is unigraphic. $b_m > 0$, hence $m \geqslant 2$. If $m = 2$, then $\psi = (1, 1)$ and if $m = 3$, then $\psi = (2, 1, 1)$ or $\psi = (2, 2, 2)$. In each case $\delta(1, \psi) = 0$.

Suppose $m \geqslant 4$ and $b_2 > b_{m-1}$. Then, by Theorem 3.2, there exists j, $1 \leqslant j < m$, such that $\delta(j, \psi) = 0$.

To complete the proof of the Theorem it is enough to show that the assumption $\delta(j, \psi) = 0$ for some j, $1 \leqslant j < m$ leads to a contradiction.

Suppose that $\delta(j, \psi) = 0$ for some j, $1 \leqslant j < m$. Fix such j. Let $G(p_1, \ldots, p_n)$ be a realization of ϕ. Define $S_1 = \{p_1, \ldots, p_{j_0}\}$, $T_1 = \{p_{h'+1}, \ldots, p_n\}$, $U_1 = \{p_{j_0+1}, \ldots, p_{h'}\}$. S_1, T_1, U_1 fulfill conditions (2.1) to (2.4), by Lemma 2.1. Define $h'' = h(j)$ (in ψ), $S_2 = \{p_{j_0+1}, \ldots, p_{j_0+j}\}$, $T_2 = \{p_{j_0+h''+1}, \ldots, p_{h'}\}$, $U_2 = \{p_{j_0+j+1}, \ldots, p_{j_0+h''}\}$. By Lemma 2.1, S_2, T_2, U_2 (in $G[U_1]$) fulfill conditions (2.1) to (2.4). Define $S = S_1 \cup S_2$, $T = T_1 \cup T_2$, $U = U_2$. It is easy to check that S, T and U also fulfill conditions (2.1) to (2.4). Hence, by Lemma 2.2, $\delta(j_0 + j, \phi) = 0$, a contradiction to the maximality of j_0.

Up to now we assumed always $a_2 > a_{n-1}$. In the next two sections we will abandon this assumption.

4. The Regular Case

Definition 4.1. ϕ^c, the complementary sequence of ϕ, is defined by $\phi^c = (a_1{}^c, \ldots, a_n{}^c) = (n-1-a_1, \ldots, n-1-a_n)$.

Remark. The complementary graph of any realization of ϕ, is a realization of ϕ^c. Hence ϕ is unigraphic iff ϕ^c is unigraphic.

Theorem 4.1. *Suppose $a_1 = a_n$. Then ϕ is unigraphic iff $a_1 \in \{1, n-2, n-1\}$ or $\phi = (2, 2, 2, 2, 2)$.*

Proof. We may assume $a_1 \leqslant (n-1)/2$, since otherwise $a_1{}^c < (n-1)/2$.

Case a. $1 < a_1 < (n-1)/2$. Let $\phi_1 = (a_1, \ldots, a_{a_1+1})$, $\phi_2 = (a_{a_1+2}, \ldots, a_n)$. Both ϕ_1 and ϕ_2 are graphic and hence ϕ has a disconnected realization. On the other hand, $\sum_{i=1}^n a_i = na_1 > 2(n-1)$ and therefore ϕ has a connected realization (see [1; 2, (1.6)]).

Case b. $a = (n-1)/2$. In this case n is odd, hence a is even. Let $a = 2m$. Choose a polytope on n vertices, $p_1, \ldots, p_n$ (in this order). By adding to the polytope all the diagonals with "length" $<m$ we obtain a graph $G = G(p_1, \ldots, p_n)$ which realizes ϕ. (Where the length of a diagonal is the number of edges of the polytope, on the shortest path between the endpoints of the diagonal). No set of $m+2$ vertices form in G a clique (i.e., for no set $Y = \{p_{i_1}, \ldots, p_{i_{m+2}}\}$, does $G[Y]$ equal k_{m+2}).

Let $X = \{p_2, p_3, \ldots, p_{m+3}\}$. $G[X] = k_{m+2} - (p_2, p_{m+3})$. If $a_1 > 2$ $(m > 1, n > 9)$, then $(p_1, p_{m+4}) \notin G$. Hence $G^* = G - (p_1, p_2) - (p_{m+3}, p_{m+4}) + (p_1, p_{m+4}) + (p_2, p_{m+3})$ realizes ϕ. G^* is not isomorphic to G since $G^*[X] = K_{m+2}$.

5. The Almost Regular Cases

Lemma 5.1. *If* $n - 1 > a_1 > a_2 \geqslant \cdots \geqslant a_{n-1} > a_n$, *then* ϕ *is not unigraphic.*

Proof. Using Theorem 3.1 twice we obtain a realization $G_1(p_1, ..., p_n)$ of ϕ and a realization $G_n(p_1, ..., p_n)$ of ϕ. In G_1, p_1 is adjacent to $p_2, p_3, ..., p_{a_1+1}$, and hence p_1 is not adjacent to p_n. In G_n, p_n is adjacent to $p_1, p_2, ..., p_{a_n}$. Therefore G_1 and G_n are not isomorphic.

Lemma 5.2. (a) *If* $a_1 > a_2 = a_n$, *then* ϕ *is unigraphic iff* $(a_2 - 1, a_3 - 1, ..., a_{a_1+1} - 1, a_{a_1+2}, ..., a_n)$ *is unigraphic.*

(b) *If* $a_1 = a_{n-1} > a_n$, *then* ϕ *is unigraphic iff* $(a_1 - 1, ..., a_{a_n} - 1, a_{a_n+1}, ..., a_{n-1})$ *is unigraphic.*

Proof. The Lemma is self-evident.

In light of Lemma 5.2 we will investigate, in the next two lemmas, sequences of the form $(a, ..., a, a - 1, ..., a - 1)$.

Lemma 5.3. *If* $n - 1 > a_1 = a_k = a_{k+1} + 1 = a_n + 1$ *and* $2 \leqslant k \leqslant n - 2$, *then*

(a) $\delta(j, \phi) > 0$ *for* $j \neq k$.

(b) $\delta(k, \phi) = 0$ *iff* $\phi = (2, 2, 1, 1)$.

Proof. If $\delta(j, \phi) = 0$ for some j $(1 \leqslant j < n)$, then $a_n < j$, since otherwise:

$$j(n - j) = j(n - 1) - j(j - 1) > \sum_{i=1}^{j} a_i - j(j - 1)$$
$$= \sum_{i=j+1}^{n} \min(j, a_i) = j(n - j), \quad \text{a contradiction.}$$

But $a_1 > j - 1$, since otherwise

$$0 \geqslant \sum_{i=1}^{j} a_i - j(j - 1) = \sum_{i=j+1}^{n} \min(j, a_i) \geqslant a_n > 0, \quad \text{a contradiction.}$$

Hence $j \geqslant a_1 > j - 1$, or $a_1 = j$.

Suppose $\delta(j, \phi) = 0$ for $j < k$. Then

$$j = j^2 - j(j - 1) = \sum_{i=1}^{j} a_i - j(j - 1) = \sum_{i=j+1}^{n} \min(j, a_i)$$
$$= j(k - j) + (j - 1)(n - k) > j, \quad \text{a contradiction.}$$

$(j - 1 = a_1 - 1 \geqslant 1, n - k > 1)$.

Suppose $\delta(j, \phi) =$ for $n > j > k$. Then $k = \sum_{i=1}^{j} a_i - j(j-1) = (j-1)(n-j) \geqslant j - 1 \geqslant k$. Hence $n - j = 1$, or $n - a_1 = 1$, a contradiction.

Suppose $\delta(k, \phi) = 0$. Then

$$k = \sum_{i=1}^{k} a_i - k(k-1) = (k-1)(n-k). \tag{5.1}$$

$n - k \geqslant 2$, hence $k \geqslant 2(k-1)$, or $2 \geqslant k$. Since $k \leqslant 2$ we have $a_1 = k = 2$ and from condition (5.1), $n = 4$.

LEMMA 5.4. *If* $\psi = (b,..., b, b-1,..., b-1)$, *where* b *occurs* $k \geqslant 2$ *times and* $b - 1$ *occurs* $n - k \geqslant 2$ *times, then* ψ *is unigraphic iff one of the following holds*:

(a) $\psi = (2, 2, 1, 1)$.

(b) $\psi = (1,..., 1, 0,..., 0)$ (*and* k *is even*).

(c) $\psi = (n-1,..., n-1, n-2,..., n-2)$ (*and* $n - k$ *is even*).

Proof. If $n - 1 > b > 1$, then, by Theorem 3.2, ψ is unigraphic only if $\delta(j, \psi) = 0$ for some $1 \leqslant j < n$. By Lemma 5.3, $\delta(j, \psi) = 0$ only if $\psi = (2, 2, 1, 1)$ (and $j = 2$). The sequence (2, 2, 1, 1) is of course unigraphic. If $b = 1$ then ψ is unigraphic for any even k. If $b = n - 1$, then $\psi^c = (0,..., 0, 1,..., 1)$ and hence ψ^c and ψ are unigraphic for any even $n - k$.

From Lemmas 5.1 to 5.4 we obtain, in the next Lemma, a characterization for unigraphic sequences in which $a_2 = a_{n-1}$.

LEMMA 5.5. *Let* $\psi = (b_1,..., b_n)$ *be a graphic sequence*, $n - 1 > b_i > 0$ $(i = 1,..., n)$ *and* $b_1 \neq b_2 = b_3 = \cdots = b_n$. *Then* ψ *is unigraphic iff one of the following conditions holds*:

(1) $b_2 = 1$

(2) $b_1 = n - 2$ *and* $b_2 = 2$

(3) $b_2 = n - 2$

(4) $b_1 = 1$ *and* $b_2 = n - 3$

Proof. If condition (3) or condition (4) holds for ψ, then condition (1) or condition (2), respectively, holds for ψ^c. Hence we may assume that $b_1 > b_2$.

By Lemma 5.2, ψ is unigraphic iff $\psi_1 = (b_2,..., b_k, b_{k+1} - 1,..., b_n - 1)$ is unigraphic, where $k = n - b_1$. $n - 1 > b_1 > 1$, hence $n - 2 \geqslant k \geqslant 2$.

Case a. $b_1 < n - 2$. In this case $k > 2$ and we may apply Lemma 5.4 to ψ_1. $\psi_1 \neq (2, 2, 1, 1)$. Since otherwise $\psi = (2,..., 2)$. $n - 1 > b_1 > b_2$, hence $b_2 < (n - 1) - 1$; thus $\psi_1 \neq ((n - 1) - 1,..., (n - 1) - 1, (n - 1) - 2,..., (n - 1) - 2)$. Therefore ψ_1 is unigraphic iff $\psi_1 = (1,..., 1, 0,..., 0)$ and $\psi = (b_1, 1,..., 1)$.

Case b. $b_1 = n - 2$, or $k = 2$. In this case $\psi_1 = (b_2, b_3 - 1,..., b_n - 1)$. $b_2 < (n - 1) - 2$, since the sequence $(n - 2, n - 3,..., n - 3)$ is not graphic. Thus we may apply case a to ψ_1. Therefore $\psi_1 = (b_2, 1,..., 1)$. Hence $b_2 = b_3 = 2$ and $\psi = (n - 2, 2,..., 2)$.

6. Conclusions

In this short section we summarize our results into one main theorem:

Theorem 6.1 (Main Theorem). *Given a sequence* ϕ, $n > 3$, *let* j_0 *be the largest index* j, *such that* $\delta(j, \phi) = 0$, $(0 \leqslant j \leqslant n)$; *and let* $h = h(j_0)$ (*see Definition* 2.1).

If $a_2 = a_{n-1}$, *then* ϕ *is unigraphic if and only if one of the following conditions holds*:

$$a_1 = a_n, \qquad a_n \in \{1, n - 2, n - 1\} \tag{6.1}$$

$$a_1 = a_n = 2, \qquad n = 5 \tag{6.2}$$

$$a_1 > a_2 = a_n = 1 \tag{6.3}$$

$$a_1 > a_2 = a_n = 2, \qquad a_1 \in \{n - 1, n - 2\} \tag{6.4}$$

$$n - 2 = a_1 = a_{n-1} > a_n \tag{6.5}$$

$$n - 3 = a_1 = a_{n-1} > a_n = 1 \tag{6.6}$$

$$n - 1 = a_1 > a_2 = a_n = 3, \qquad n = 6 \tag{6.7}$$

$$n - 1 = a_1 > a_2 = a_n = n - 2. \tag{6.8}$$

If $a_2 > a_{n-1}$, *then* ϕ *is unigraphic if and only if*

(1) *either* $h = n$, *or the pair* $[(a_1 - (h - 1),..., a_{j_0} - (h - 1)), (a_{h+1},..., a_n)]$ *is unigraphic, and*

(2) *either* $h = j_0$, *or* $h > j_0 + 3$ *and the sequence* $(a_1',..., a_{n'}') = (a_{j_0+1} - j_0,..., a_h - j_0)$ *fulfills one of the conditions* (6.1)-(6.6).

Proof. The Theorem follows easily from Lemma 2.4, Theorem 3.2, Theorem 3.3, Theorem 4.1, Theorem 5.1, and Lemma 5.5.

Remark. While writing this paper, the author learned that Shuo-Yen R. Li wrote a paper on the same subject [6]. In particular Theorem 4.1 and Lemma 5.2 appear in [6] too. However, the attitude and the nature of his results are different from ours.

References

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